The Illusion of Thin-Tails Under Aggregation

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Abstract: It is assumed that while portfolio theory fails with daily returns, that it would work with yearly returns, an standard argument recently repeated in Treynor (2011). This paper debunks the confusion that daily returns, when nonGaussian but with finite variance can aggregate to thin tails. Alas, portfolio theory fails in both the short and the long run. The central limit theorem operates too slowly for economic data for us to use it and take portfolio theory with any degree of seriousness. The point is illustrated with a Monte Carlo simulation.

We disagree with the statements in Treynor (2011) on logical, epistemological, empirical, and mathematical grounds.

Let us summarize Treynor's key argument: he accepts that equity market returns on a daily basis are fat-tailed, in agreement with the first author. But he then argues that such returns, by aggregation, become thin tailed —in other words, when we look at yearly, not daily, data. To the question "do annual returns behave as Taleb would predict [sic] correlated, with dispersion that shifts too rapidly to have a finite variance?", he offers a stark "no"—offering as evidence the past 195 years of annual equity market returns which seem to disconfirm such theory. In sum, what he offers is N=195 years of annual returns as a test of his theory of aggregation (which is, no doubt, afflicted by the survivorship bias associated with a continuous financial market that has existed while many have failed but we can ignore the point for now).

Alas, we believe that the data that he produces is not just entirely compatible with presence of fat tails under aggregation, but what one should expect, as small sample effects may mask the tail effects, *regardless* of whether or not the data has infinite or finite variance. In particular, Treynor (and others) incorrectly rely on the central limit theorem, which states that the distribution of the sum of scaled random variables independently drawn from a finite-variance distribution will converge to a Gaussian distribution. However, the central limit theorem is only valid *asymptotically*, and in finite samples, the *center* of the distribution converges towards a Gaussian distribution before the *tails.* This is a repeat of the mistake made by officer (1972), debunked in Taleb (2009a) which showed how Kurtosis fails to decline under aggregation, once one is aware of the small sample effect.

Let us define a power-law as, for extreme values, the ratios of exceedance probabilities are scale invariant. So take X a random variable, we have a power-law if for x large enough, $P[X>x] \sim O[x^{\alpha}]$.

For aggregation of random variables drawn from distributions with power law tails (and finite variance) of, say, a power law exponent α of 3, the width of the Gaussian component is proportional to $\sigma N^{1/2}(\ln N)^{1/2}$ (Bouchaud and Potters, 2003; Sornette, 2004). The growth in width can be thought of deriving from two components – that inherent growth of the standard deviation of the sum from addition of independent random variables—which constitutes the body of the distribution--and then a factor of $(\ln N)^{1/2}$ that represents any taming of the tails. Further, using extreme value theory, the maximum of a random variable is not affected much by aggregation as the aggregation effect is very slow, even in the presence of finite variance (Mandelbrot and Taleb, 2010).

Clearly we have had for sometime —since Mandelbrot (1963) —sufficient evidence that daily returns in finance and commodities markets are power-law distributed, or, to say the least, insufficient evidence to *reject* it; this fact has been particularly rejuvenated by research conducted by the "Econophysics" groups. Almost all empirical results describe returns even when they have finite variance, as power-law distributed (Sornette, Gabaix et al.,2003, Stanley et al.,2000). Further, Weron (2001) showed that infinite variance processes can masquerade as having an $\alpha>2$, again, from small sample effects.

In part because of the influence of research on non-Gaussian stable distributions, it seemed crucial that fat tails were paired with the focus on values of $\alpha<2$, where the variance is "infinite". But Taleb (2009b) showed that the requirement for power-law effects such as fat tails is not the same requirement of finite or infinite variance: power-laws are power-laws, and their mere acceptance invalidates the results of Markowitz (1959) portfolio theory. Simply, power laws have infinite moments higher than α, meaning that even if variance is the highest moment needed for the Markowitz derivations, this is merely an asymptotic property that breaks down anywhere before infinity. The results of Markowitz cannot accommodate power laws, finite or infinite variance (though derivatives pricing is not affected at all by such argument).

The empirical tests in the literature point to a "cubic" power-law, with an α around 2.7.

Let us look at the properties of aggregated (log) returns and determine whether the return properties in Treynor (2011) may be consistent with fat tails at the weekly level or the annual level.

Let x be daily log returns, and take x randomly generated following the distribution with tail exponent α= 2.7

$$\frac{\left(\frac{\alpha}{x^2+\alpha}\right)^{\frac{1+\alpha}{2}}}{\sqrt{\alpha}\ \text{Beta}\left[\frac{\alpha}{2},\frac{1}{2}\right]}$$

we take the sum to get y, the annual return,

$$y_{j,z}=\sum_{i=1}^{252} x_{i,z}$$

and

$$Y_z=\left\{y_{j,z}\right\}_{j=1}^{195}$$

We calculate the kurtosis of the distribution of annual returns for a given run z

$$K(z)=\frac{\frac{1}{195}\sum_{j=1}^{195}\left(y_{j,z}-\bar{y}_z\right)^4}{\left(\frac{1}{195}\sum_{j=1}^{195}\left(y_{j,z}-\bar{y}_z\right)^2\right)}$$

Finally, we end with the vector of length M

$$\left\{K(z)\right\}_{z=1}^{M}$$

Now by standard result, the expectation of K(z) is infinite. But small (by which we mean finite) samples may yield a different result, which may obscure the unboundness of the kurtosis for this distribution. We therefore conduct a sampling exercise.

We generated close to 4 billion, 3.82 10^7 daily runs of x, for a total M of 78 10^3. The median kurtosis observed is 3.36. For a purely Gaussian process the median kurtosis observed for a sample of 200 is 2.95; the probability of observing a kurtosis of 3.36 for a sample size of 200 is approximately .88. Thus, even with infinite higher moments, we may find that observed moments may be consistent with the moments from "thin" tailed distributions.

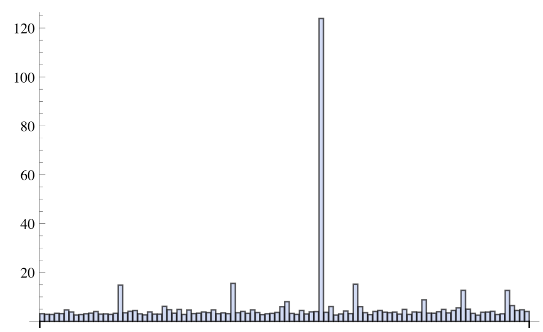

Figure 1— The "small" Monte Carlo run illustrating the distribution of Kurtosis, here M=100.

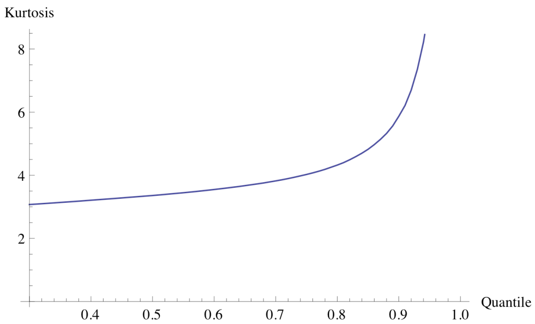

Figure 2, Distribution M=78,000 from 3.8 billion simulations of daily returns x

Data: $3.8\ 10^7$
Median: 3.35
Mean:4.8231
Quantile (.98): 19.94
Max: 189.84

What is one of many the lesson from this exercise? Moment-based empirical reasoning is an unreliable basis for determining the distributional properties of the underlying return generating process. If our concern is risk, then we should focused on tail-regarding measures of risk, rather than take misplaced comfort in moment-based measures.

We note further, that for an N of 52, we have a width that is 1.3 times the aggregate standard deviation, thus indicating even for finite variance distributions, under power-laws the rate at which tails become Gaussian is very slow relative to sample sizes. Even taking 195x52 weeks of data, or more than 10,000 observations, the Gaussian component of the distribution is only 2.0 times the aggregate standard deviation. Assuming aggregation of daily data, we have the Gaussian component of the distribution through 2.16 times the aggregate standard deviation.

Conclusion

Treynor (2011) argues from historical data and an imprecise interpretation of the central limit theorem that equity market return data is, when aggregated, sufficiently well behaved to employ volatility-based risk measures derived from price returns. However, we argue that the line of reasoning he pursues – namely misplaced reliance on the central limit theorem – and the evidence he presents (195 annual returns for equity markets), do not in any way contradict the presence of fat-tailedness in financial market returns, even when aggregated to be considered annually. Moreover, we argue that moment-based measures of risk are insufficient for, particularly for passive/static exposures to, financial market assets.

The harder conclusion is that portfolio theory (and other results flowing from it) are alas unusable in the real world, whether for daily, monthly, annual, or centennial returns .

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